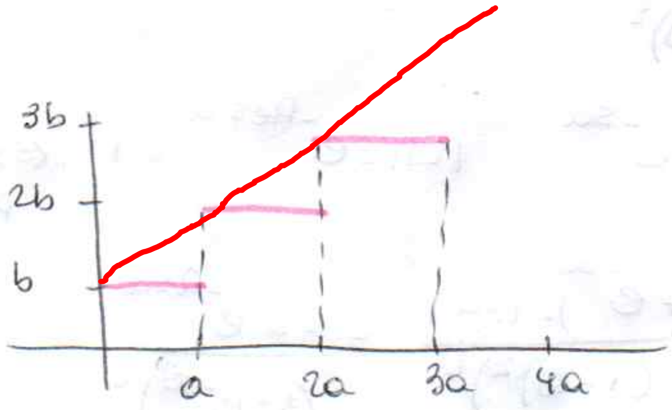
T. Laplace

Ejercicio 3, TP8.

$$f(t) = \begin{cases} nb & \text{si } (n-1)a \le t < na \quad n \in \mathbb{N} \quad a,b>0 \\ 0 & \text{si } t<0 \end{cases}$$



orden exponencial?

$$|f(t)| \le b + t\frac{b}{a} \le b + Me^{\alpha t} \le (b+M)e^{\alpha t}$$

para $\alpha > 0$

Su transformada:

$$\int_0^\infty f(t)e^{-st}\,dt = \lim_{N\to\infty} \int_0^{Na} f(t)e^{-st}\,dt = \lim_{N\to\infty} \sum_{n=1}^{N} \int_{(n-1)a}^{na} nbe^{-st}\,dt$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} nb\frac{e^{-st}}{-s}\Big|_{(n-1)a}^{na} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{nb}{s}\left(e^{-s(n-1)a} - e^{-sna}\right) =$$

$$= \sum_{n=1}^{\infty} \frac{nb}{s} e^{-sna}(e^{sa}-1) = \frac{(e^{sa}-1)}{s}.b.\sum_{n=1}^{\infty} n(e^{-sa})^n$$

Como $\sum_{0}^{\infty} \square^n = \frac{1}{1-\square}$ si $|\square|<1$

Entonces $\sum_{1}^{\infty} n.\square^n = \sum_{1}^{\infty} n.\square^{n-1}.\square = \square\left(\sum_{0}^{\infty}\square^n\right)' = \square.\left(\frac{1}{1-\square}\right)'$

$$= \square . \frac{1}{(1-\square)^2}$$

En este caso tenemos $\square = e^{-sa}$ $\quad |\square| = e^{-\mathrm{Re}\,s\,a} < 1 \iff \mathrm{Re}\,s > 0$

$$\Rightarrow \sum_{1}^{\infty} n\left(e^{-sa}\right)^n = e^{-sa} . \frac{1}{(1-e^{-sa})^2}$$

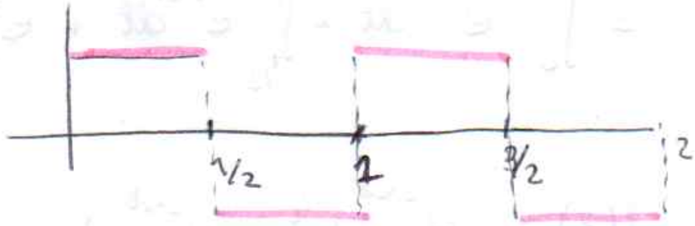
Entonces;

$$\mathcal{L}(f)(s) = \frac{(e^{sa}-1).b}{s} \frac{e^{-sa}}{(1-e^{-sa})^2} = \frac{e^{sa}(1-e^{-sa})\,b\,e^{-sa}}{(1-e^{-sa})^2} =$$

$$\boxed{= \frac{b}{s(1-e^{-sa})}} = \frac{b\,e^{sa}}{s(e^{sa}-1)}$$

Notar que si a=1,b=1, la función dada es f(t) = [t+1] (parte entera de t+1), y la función parte entera está transformada en clase 29, páginas 11 y 12. Relacionen la transformada aquí obtenida con la transformada de parte entera.

5.a) $f(t) = \begin{cases} 1 & 0 \leq t \leq 1/2 \\ -1 & 1/2 \leq t < 1 \end{cases}$



$\mathcal{L}(f) = \int_0^\infty f(t) e^{-st} dt =$

$= \lim\limits_{N \to \infty} \int_0^N f(t) e^{-st} dt = \lim\limits_{N \to \infty} \sum\limits_{n=1}^{N} \int_{n-1}^{n} f(t) e^{-st} dt =$

$\lim\limits_{N \to \infty} \sum\limits_{n=1}^{N} \int_{n-1}^{n-1/2} 1 e^{-st} dt + \int_{n-1/2}^{n} (-1) e^{-st} dt =$

$\lim\limits_{N \to \infty} \sum\limits_{n=1}^{N} \left(\frac{e^{-st}}{-s} \Big|_{n-1}^{n-1/2} - \frac{e^{-st}}{-s} \Big|_{n-1/2}^{n} \right) =$

$\sum\limits_{n=1}^{\infty} \frac{e^{-sn} e^{s/2} - e^{-sn} e^{s}}{-s} - \frac{e^{-sn} - e^{-sn} e^{s/2}}{-s} =$

$\sum\limits_{n=1}^{\infty} \frac{e^{-sn}}{-s} \underbrace{\left(e^{s/2} - e^{s} - 1 + e^{s/2} \right)}_{-(e^{s} - 2e^{s/2} + 1)} = \frac{(e^{s/2} - 1)^2}{s} \sum\limits_{1}^{\infty} e^{-sn} =$

Como: $\sum\limits_{1}^{\infty} \square^n = \sum\limits_{0}^{\infty} \square^n - 1 = \frac{1}{1-\square} - 1 = \frac{\square}{1-\square}$ si $|\square| < 1$

$\mathcal{L}(f) = \frac{(e^{s/2} - 1)^2}{s} \cdot \frac{e^{-s}}{(1 - e^{-s})}$ si $|e^{-s}| < 1$ $(\operatorname{Re} s > 0)$

Otra forma (según visto en teórico:)

$\mathcal{L}(f) = \int_0^\infty f(t) e^{-st} dt = \int_0^1 f(t) e^{-st} dt + \int_1^\infty f(t) e^{-st} dt = \cancel{\int_0^1 f(t) e^{-st} + \int_0^\infty f(u+1)} e^{-s(\ldots}$

$u = t - 1$

$= \int_0^1 f(t) e^{-st} dt + \int_0^\infty f(u+1) e^{-s(u+1)} dt =$

$$= \int_0^{1/2} e^{-st}\,dt - \int_{1/2}^{1} e^{-st}\,dt + e^{-s}\underbrace{\int_0^{\infty} f(u)e^{-su}\,du}_{\mathcal{L}(f)}$$

$$\mathcal{L}(f) = \frac{e^{-st}}{-s}\Big|_0^{1/2} - \frac{e^{-st}}{-s}\Big|_{1/2}^{1} + e^{-s}\mathcal{L}(f)$$

$$\mathcal{L}(f)(1-e^{-s}) = \frac{e^{-s/2}-1}{-s} - \frac{(e^{-s}-e^{-s/2})}{-s}$$

$$\mathcal{L}(f)(s) = \frac{1}{s(1-e^{-s})}\left(e^{-s}-2e^{-s/2}+1\right) = \frac{e^{-s}(1-2e^{s/2}+e^{s})}{s(1-e^{-s})}$$

$$= \frac{e^{-s}(1-e^{s/2})^2}{s\cdot(1-e^{-s})} = \frac{1}{s(1-e^{-s})}\cdot(e^{-s/2}-1)^2$$